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# Examples of Functions Whose Sequence of Strong Unicity Constants Is Unbounded

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## 1. INTRODUCTION

Let $C(I)$ denote the space of all real-valued continuous functions on the interval $I = [-1, 1]$ with the uniform norm $\|\cdot\|$. Let $\Pi$ denote the set of all algebraic polynomials and $\Pi_n \subset \Pi$ the set of all algebraic polynomials of degree at most $n$. It is known [6] that for $f \in C(I)$ there exists a unique $B_n(f) \in \Pi_n$ and a positive constant $\gamma$ such that

$$\|f - p\| \geqslant \|f - B_n(f)\| + \gamma \|p - B_n(f)\| \tag{1.1}$$

for all $p \in \Pi_n$. The polynomial $B_n(f)$ is called the best approximation of $f$ from $\Pi_n$ and the largest constant $\gamma$ satisfying (1.1) is called the strong unicity constant. This constant depends on both the function $f$ and the integer $n$ and will be denoted by $\gamma_n(f)$. In this paper it will be more convenient to consider the reciprocal of $\gamma_n(f)$ and we will use the notation

$$M_n(f) = [\gamma_n(f)]^{-1} \geqslant 1.$$

The behavior of the sequence $\{M_n(f)\}_{n=0}^{\infty}$ has been the subject of a number of investigations [1, 4, 5, 8] which are directed at the resolution of a question that was first posed by Poreda [7]: For what $f \in C(I)$ is the sequence $\{M_n(f)\}$ bounded?

Define the set

$$\mathscr{B} = \{f \in C(I): \overline{\lim_n}\, M_n(f) < \infty\}$$

and note that $\Pi \subseteq \mathscr{B}$. Henry and Roulier [5] have conjectured that the reverse inclusion also holds. A survey of the previously mentioned results supports their conjecture and shows that the behavior of $\{M_n(f)\}$ depends on the cardinality of the extreme set of $f$.



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We denote the extreme sets of $f \in C(I)$ by

$$E_n(f) = \{x \in I: |f(x) - B_n(f)(x)| = \|f - B_n(f)\|\}$$

and the cardinality of $E_n(f)$ by $|E_n(f)|$. The classical Tschebyscheff Equioscillation theorem [3] asserts that $|E_n(f)| \geqslant n+2$. Schmidt [8] has shown that if $|E_n(f)| = n+2$ for infinitely many $n$ then $f \notin B$. This raised the question of whether or not there exists non-polynomial $f \in C(I)$ which has the property that $|E_n(f)| > n+2$ for all but finitely many values of $n$. Bartelt and Schmidt [1] settle this existence question in the affirmative by appealing to the Baire Category theorem and as such the method is not constructive. In the present work a class of functions is constructed whose extreme sets contain more than $n+2$ points for all sufficiently large $n$. In particular, the function $S_{2k}$ considered in Section 2 has the property that, for a given positive integer $k$,

$$|E_n(S_{2k})| \geqslant n + k + 2.$$

Bartelt and Schmidt [1] obtain a sufficient condition for a function to be in the complement of $\mathscr{B}$. Specifically, they establish the following theorem.

THEOREM 1. *Let $f \in C(I)\backslash \Pi$. If $|E_n(f)| \leqslant n+4$ for all sufficiently large $n$, then $f \notin \mathscr{B}$.*

In Section 4, Theorem 1 is strengthened for the class of even functions and the results of Section 4 are used to show the unboundedness of the sequence $\{M_n\,|x|\}_{n=0}^{\infty}$. We note that $|x| \notin C^1(I)$ and the results in [5, 8], which require $f$ to be in $C^{\infty}(I)$, are not applicable to the absolute value function.

## 2. Large Extreme Sets

In this section we exhibit a function $f \in C(I)$ whose extreme sets contain at least $n+k+2$ points where $k$ is an arbitrary fixed positive integer. The method is constructive and the resulting function can be seen to have symmetry properties similar to the symmetry of an even function.

We define the error function for $f$ as

$$r_n(f) = f - B_n(f). \tag{2.1}$$

A set of points $x_1 < x_2 < \cdots < x_N$ will be called an alternation set for the function $r_n(f)$ if $r_n(f)(x_i) = -r_n(f)(x_{i+1}) = \pm \|r_n(f)\|$ for $i = 1,\dots, N-1$. The Tschebyscheff Equioscillation theorem [3] asserts that $r_n(f)$ has an alternation set with cardinality at least $n+2$, and conversely, if $r = f - p$

where $p \in \Pi_n$ has an alternating set with cardinality at least $n+2$ then $p = B_n(f)$.

The Tschebyscheff polynomial of degree $k$ on $I$ will be denoted by $T_k$. For $k > 0$, the range of $T_k$ is $I$ and thus for any $g \in C(I)$, $\|g\| = \|g \circ T_k\|$. The construction of an $f \in C(I)$ with $|E_n(f)| \geqslant n + k + 2$ requires the following lemmas.

LEMMA 1. *Let* $h \in C(I)$, $f = h \circ T_k$, *and* $l = nk + m$, $m = 0, 1, \ldots, k-1$, where *$n$ and $k$ are positive integers, then*

$$\text{(i)} \quad B_{nk}(f) = B_n(h) \circ T_k = B_l(f)$$

*and*

$$\text{(ii)} \quad r_{nk}(f) = r_l(f) \text{ exhibits } (N-1)k + 1 \text{ alternations}$$

*if* $r_n(h)$ *exhibits $N$ alternations.*

*Proof.* We denote by $\{x_j\}_{j=1}^N$ an alternation set for $r_n(h)$ where we can assume that the $x_i$ have been ordered as follows:

$$-1 \leqslant x_N < x_{N-1} < \cdots < x_1 \leqslant 1.$$

Denote by $\{y_i\}_{i=0}^k$ the $k+1$ points where $T_k(y_i) = (-1)^i$. Without loss of generality we may assume that

$$-1 = y_k < y_{k-1} < \cdots < y_0 = 1.$$

The restriction of $T_k$ to the interval $[y_i, y_{i-1}]$ $(i = 1, \ldots, k)$ is a bijection onto $I$. We define $x_{ij}$ to be the unique element of $[y_i, y_{i-1}]$ with the property that $T_k(x_{ij}) = x_j$ $(i = 1, \ldots, k;\ j = 1, \ldots, N)$. Define the set $X = \{x_{ij}: i = 1, \ldots, k;\ j = 1, \ldots, N\}$. The ordering of $X$ is most conveniently described as follows: fix $i$, then $x_{ij}$ is strictly decreasing (increasing) if $i$ is odd (even). Reference to Fig. 1 will be beneficial in describing the rest of the ordering of $X$. Equality may occur in the transition from one row to the next (the arrows of Fig. 1) only in the first and/or last columns.

$$\begin{array}{l} x_{11} > x_{12} > \cdots > x_{1N} \searrow \\ \swarrow x_{21} < x_{22} < \cdots < x_{2N} \swarrow \\ \searrow x_{31} > x_{32} > \cdots > x_{3N} \searrow \\ \qquad\qquad \vdots \end{array}$$

FIGURE 1

For example, we must have $x_{iN} \geqslant x_{i+1,N}$ where equality occurs if and only if $i$ is odd and $-1 \in E_n(h)$. Analogously, $x_{i1} \geqslant x_{i+1,1}$ with equality occurring if and only if $i$ is even and $1 \in E_n(h)$.

We define $r = f - B_n(h) \circ T_k$ and from the definition of $f$ we have $r = (h - B_n(h)) \circ T_k = r_n(h) \circ T_k$. Since $\{x_j\}_{j=1}^N$ is an alternation set for $r_n(h)$ and $T_k(x_{ij}) = x_j$ it follows that, for each $i = 1,..., k$, the set $\{x_{i1},..., x_{iN}\}$ is an alternation set for $r$. Moreover, we note that $r(x_{2i,1}) = h(x_1) - B_n(h)(x_1) = r(x_{2i+1,1})$ for $i = 1,..., [(k+1)/2]$ and $r(x_{2i-1,N}) = r(x_{2i,N})$ for $i = 2,..., [(k+1)/2]$. It follows that the set

$$A = \{x_{ij}: i = 1,..., k; j = 2,..., N-1\}$$
$$\cup \left\{x_{2i-1,j}: i = 1,..., \left[\frac{k+1}{2}\right]; j = 1, N\right\} \cup Y$$

where $Y$ is the singleton $\{x_{k1}\}$ if $k$ is even and empty if $k$ is odd, is an alternation set for $r$. The cardinality of $A$ is given by

$$|A| = (N-2)\,k + 2\left[\frac{k+1}{2}\right] + |Y| = (N-1)\,k + 1.$$

The Tschebyscheff Equioscillation theorem implies that $N \geqslant n+2$. Since $nk = l - m$ and $0 \leqslant m \leqslant k-1$, we have

$$\begin{aligned} |A| &= (N-1)\,k + 1 \geqslant nk + k + 1 \\ &= l - m + k + 1 \\ &\geqslant l + 2. \end{aligned}$$

Thus $r = f - B_n(h) \circ T_k$ exhibits at least $l+2$ alternations and $B_l(f) = B_n(h) \circ T_k$ is the best approximation to $f$ from $\Pi_l$. This establishes (i). Furthermore, $r = r_{nk}(f) = r_l(f)$ and (ii) follows.

LEMMA 2. *Let the functions $f$, $h$, and $T_k$ and the integers $n$, $k$, $l$, and $m$ be as in Lemma 1. Then $E_l(f) = T_k^{-1}(E_n(h))$.*

*Proof.* By Lemma 1, $f - B_l(f) = (h - B_n(h)) \circ T_k$ so that $x \in E_l(f)$ if and only if $T_k(x) \in E_n(h)$.

LEMMA 3. *Let the functions $f$, $h$, and $T_k$ and the integers $n$, $k$, $l$, and $m$ be as in Lemma 1. If $|E_n(h)| = N \geqslant n+2$ then the cardinality of $E_l(f)$ has the following lower bounds:*

(i) $|E_l(f)| = kN \geqslant l + k + 1$ if $\pm 1 \notin E_n(h)$,

(ii) $|E_l(f)| = k(N-1) + \left[\frac{k}{2}\right] + 1 \geqslant l + \left[\frac{k}{2}\right] + 2$ if $1 \in E_n(h)$ and $-1 \notin E_n(h)$,

$$\text{(iii)}\quad |E_l(f)| = k(N-1) + \left[\frac{k-1}{2}\right] + 1 \quad \text{if } 1 \notin E_n(h) \text{ and}$$

$$\geqslant l + \left[\frac{k-1}{2}\right] + 2 \qquad -1 \in E_n(h).$$

$$\text{(iv)}\quad |E_l(f)| = k(N-1) + 1 \geqslant l + 2 \qquad \text{if } \pm 1 \in E_n(h).$$

*Proof.* Using Lemma 2, we establish each of the above by counting the number of points in $T_k^{-1}(E_n(h)) = E_l(f)$. Since each of the statements are similar we record here only one of the arguments and leave the remainder to the reader. We establish (ii).

Using Fig. 1 as it applies to an extreme set and Lemma 2, $|E_l(f)| = |T_k^{-1}(E_n(h))| = k(N-1) + [k/2] + 1$. Thus the inequality $N \geqslant n+2$ and the range of $m$ implies $k(N-1) \geqslant kn + k = l + (k-m) \geqslant l+1$ or $|E_l(f)| \geqslant l + [k/2] + 2$. This establishes (ii).

We now exhibit a function $S$ which has the property that $-1 \notin E_n(S)$ for every positive integer $n$.

THEOREM 2. *Define*

$$S(x) = \begin{cases} \dfrac{1}{2}(1-x^2)^{1/2} \sin\left(\dfrac{\pi}{x+1}\right), & x \in (-1, 1], \\ 0, & x = -1, \end{cases} \tag{2.2}$$

*so that $S \in C(I)$. Then $-1 \notin E_n(S)$ for every positive interger $n$.*

*Proof.* Since $\|S\| < \frac{1}{2}$ we have $\|B_n(S)\| - \frac{1}{2} < \|B_n(S)\| - \|S\| \leqslant \|B_n(S) - S\| \leqslant \|0 - S\| = \|S\| < \frac{1}{2}$ so that

$$\|B_n(S)\| < 1. \tag{2.3}$$

Applying Markoff's inequality [2] to $B_n(S)$ and using (2.3) we obtain

$$\|B_n'(S)\| \leqslant n^2 \|B_n(S)\| < n^2. \tag{2.4}$$

For all $n \geqslant 1$ we define

$$x_n^+ = \frac{1-4n^4}{1+4n^4} \qquad \text{and} \qquad x_n^- = \frac{-(1+4n^4)}{4n^4+3}$$

and a short computation gives

$$S(x_n^+) = n^2(x_n^+ + 1) \tag{2.5}$$

$$S(x_n^-) \leqslant -n^2(x_n^- + 1). \tag{2.6}$$

We now show that $\|B_n(S)-S\| > |B_n(S)(-1)-S(-1)|$ for all positive integers $n$. This is most conveniently done by considering three cases. To simplify notation we set $P=B_n(S)$ for the remainder of the proof.

*Case* (i). If $P(-1)=0$, then $|S(-1)-P(-1)|=0$ by the definition of $S$ and $-1 \notin E_n(S)$.

*Case* (ii). If $P(-1)>0$ then the mean value theorem implies that a number $c \in (-1, x_n^-)$ may be found such that

$$\frac{P(x_n^-)-P(-1)}{x_n^- - (-1)} = P'(c) > -n^2. \tag{2.7}$$

We have used (2.4) to obtain the inequality in (2.7). We now rewrite (2.7) in the form

$$P(-1)-n^2(x_n^- + 1) < P(x_n^-). \tag{2.8}$$

Using (2.8) and (2.6) we obtain

$$\begin{aligned} P(x_n^-)-S(x_n^-) &> P(-1)-\{n^2(x_n^- + 1)+S(x_n^-)\} \\ &\geqslant P(-1)=P(-1)-S(-1)>0. \end{aligned}$$

Hence, $\|P-S\| \geqslant |P(x_n^-)-S(x_n^-)| > |P(-1)-S(-1)|$, and $-1 \notin E_n(S)$.

*Case* (iii). Here we assume that $P(-1)<0$. This is similar to case (ii). We obtain $P(x_n^+)-P(-1) < n^2(x_n^+ + 1)$ from the mean value theorem and then (2.5) gives

$$\begin{aligned} S(x_n^+)-P(x_n^+) &> \{S(x_n^+)-n^2(x_n^+ + 1)\}-P(-1) \\ &= S(-1)-P(-1)>0. \end{aligned}$$

Finally, $\|P-S\| > |S(-1)-P(-1)|$ and $-1 \notin E_n(S)$.

In all three cases $-1 \notin E_n(S)$.

It is the function $S$ discussed in Theorem 2 that enables one to construct functions $f \in C(I)$ whose extreme sets contain more than $n+2$ points.

THEOREM 3. *Let $S$ be defined as in* (2.2) *and let $k$ be a positive integer. Define $S_k = S \circ T_k$, then for all positive integers $l$*

$$|E_l(S_k)| \geqslant l+2+[k/2]. \tag{2.9}$$

*Proof.* There exist positive integers $n$ and $m<k$ such that $l=nk+m$. Theorem 2 implies that $-1 \notin E_l(S_k)$. Applying Lemma 3 to $E_l(S_k)$ gives the inequality in (2.9).

The existence of a continuous function $f$ with more than $n+2$ extreme points in [1] depended on the existence of an even continuous function with $0 \notin E_l(f)$ for all positive integers $l$ and the proof of this was not constructive. A constructive method for showing the existence of such functions is provided by Theorem 3. If we use $T_2(x) = 2x^2 - 1$ and $S$ as defined in (2.2) then Theorem 2 and Lemma 2 imply that $S_2 = S \circ T_2$ is an even function such that $0 \notin E_l(S_2)$ for all positive integers $l$. Moreover. Theorem 3 gives $|E_l(S_2)| \geqslant l+3$.

## 3. Strong Unicity of Tschebyscheff Composition

Theorem 1 provides a necessary condition for a function $f \in C(I)$ to belong to the set $\mathscr{B}$; if $f \in \mathscr{B}$ then $|E_n(f)| > n+4$. The only functions, with the latter property, known to the authors are the functions $h \circ T_k$ $(h \in C(I))$ discussed in Section 2. The construction carried out in that section cannot lead to a function in $\mathscr{B}$ unless $h \in \mathscr{B}$. This follows from Theorem 5, which uses the following characterization of $M_n(f)$ found in [1].

THEOREM 4. *If $h \in C(I) \backslash \Pi_n$ then*

$$M_n(h) = \max\{\|p\| : p \in \Pi_n, \sigma_n(h)(x)\, p(x) \leqslant 1; x \in E_n(h)\}$$

*where $\sigma_n(h)(x) = \operatorname{sgn} r_n(h)(x)$.*

THEOREM 5. *Let $h \in C(I)$ and define $f = h \circ T_k$. If $f \notin \Pi_{(n+1)k-1}$ then*

$$M_{(n+1)k-1}(f) \geqslant \cdots \geqslant M_{nk}(f) \geqslant M_n(h). \tag{3.1}$$

*Proof.* Since $f \notin \Pi_{(n+1)k-1}$, $h \notin \Pi_n$. Set $l = nk+m$ for $m = 0, 1, ..., k-1$ and let $x \in E_n(h)$. It follows from Lemma 2 that $E_l(f) = T_k^{-1}(E_n(h))$. Now for $y \in T_k^{-1}(x)$ we have

$$\begin{aligned} \sigma_l(f)(y) &= \operatorname{sgn} r_l(f)(y) \\ &= \operatorname{sgn} r_{nk}(f)(y) \\ &= \sigma_{nk}(f)(y) \\ &= \operatorname{sgn} r_n(h)(x) \\ &= \sigma_n(h)(x). \end{aligned}$$

If $p \in \Pi_n$ then $p \circ T_k \in \Pi_{nk}$ and $p \circ T_k(y) = p(x)$. Then $\Pi_{nk} \subseteq \Pi_{nk+1} \subseteq \cdots \subseteq \Pi_{(n+1)k-1}$ and Theorem 4 imply the inequalities in (3.1).

## 4. Two Examples

In this section we consider the set $\mathscr{E}$ of continuous even functions defined on $I$. It is clear that if $f \in \mathscr{E}$ then $f = \hat{f} \circ T_2$ where $\hat{f}(x) = f(\sqrt{(x+1)/2})$. Hence, the results of Section 2 are applicable to the elements of $\mathscr{E}$.

Let $f \in \mathscr{E}$ and $x = \min\{y \in E_l(f) | y \geqslant 0\}$. If $x \neq 0$ then $-x \in E_l(f)$ and $\operatorname{sgn}(r(-x)) = \operatorname{sgn}(r(x))$. Thus both $x$ and $-x$ cannot be in an alternation set. Notice that $0 \in E_l(f)$ if and only if $-1 \in E_n(\hat{f})$, $n = [l/2]$. Define $e(l) = 1$ (resp. 0) if $0 \in E_l(f)$ (resp. $0 \notin E_l(f)$). Lemma 3 implies that $|E_l(f)| = 2\,|E_n(\hat{f})| - e(l)$. Thus when $0 \in E_l(f)$ it behaves as a "double" extreme point.

Theorem 5 implies that if $f \in (\mathscr{B} \cap \mathscr{E}) \backslash \Pi$ then $\hat{f} \in \mathscr{B}$. Thus the search for a non-polynomial element of $\mathscr{B}$ should not be restricted to the even functions. Nevertheless, the following theorem shows that certain functions do not belong to $\mathscr{B}$.

Theorem 6. *Let $f \in \mathscr{E} \backslash \Pi$. If $|E_l(f)| \leqslant 2([l/2] + 4) - e(l)$, for all sufficiently large $l$, then $f \notin B$.*

*Proof.* Lemma 3 implies $|E_l(f)| = 2\,|E_n(\hat{f})| - e(l)$ where $n = [l/2]$. Thus $|E_n(\hat{f})| = \frac{1}{2}(|E_l(f)| + e(l)) \leqslant n + 4$. Theorem 1 implies $\hat{f} \notin B$ and Theorem 5 implies $f \notin B$.

Corollary 1. *If $p \in \Pi_6$ and $f(x) = p(|x|) \notin \Pi$, then $f \notin \mathscr{B}$.*

*Proof.* Let $l \geqslant 6$ and set $n = [l/2]$. Since $f$ is even we have $B_l(f)(x) = B_{2n}(f)(x) = \sum_{i=0}^{n} a_i x^{2i}$. Let $r_+$ be the restriction of the error function $r_l(f)$ to $[0, 1]$, i.e., $r_+(x) = p(x) - B_l(f)(x)$, $x \in [0, 1]$. Since $p \in \Pi_6$ and $B_l(f)(x) \in \Pi_{2n} \cap \mathscr{E}$, we have $D_x r_+ \in \Pi_{2n-1}$. Hence, $D_x r_+$ is the sum of at most $n + 3$ monomials. By Descartes' rule of signs, $D_x r_+$ has at most $n + 2$ positive roots. The non-negative elements of $E_l(f)$ must be among the roots of $D_x r_+$, zero, or one. The symmetry of $r$ now implies that $|E_l(f)| \leqslant 2n + 7 = 2[l/2] + 7$. Theorem 6 implies that $f \notin \mathscr{B}$.

The corollary implies that $f(x) = |x|$ is not an element of $\mathscr{B}$. The results in [5.8] require $C^\infty$ functions $f$ and are therefore not applicable to the absolute value function. Other examples that cannot be analyzed by considering the derivatives of $f$ can also be obtained from the methods of this paper. In particular, we show that $h(x) = (ax^2 + bx + c)\sqrt{x+1}$ does not belong to $\mathscr{B}$. Define $(h \circ T_2)(x) = f(x) = \alpha\,|x|^5 + \beta\,|x|^3 + \gamma\,|x|$. The counting argument of Corollary 1 implies that $|E_l(f)| \leqslant l + 7$, and Lemma 3 implies that $|E_n(h)| \leqslant n + 4$, $n = [l/2]$. An application of Theorem 1 yields $h \notin \mathscr{B}$.

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